Robust Finite-time stability of homogeneous systems with respect to multiplicative disturbances

Youness Braidiz, Denis Efimov, Andrey Polyakov, Wilfrid Perruquetti

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Y. Braidiz[1], D. Efimov[2], A. Polyakov[2] and W. Perruquetti[1]

***Abstract*— This paper studies the robustness properties of homogeneous finite-time stable systems with respect to multiplicative perturbation for sufficiently small inputs. Robust stability conditions are presented for the systems admitting homogeneous approximation at the origin and at infinity. The utility of the obtained results is illustrated via robustness analysis of homogeneous observer with time-varying gains.**

I. Introduction

Stability analysis and studying the robustness of the nonlinear control systems have become more and more important in the last decades (see [1]–[6]). That is why there were defined a lot of kind of robust stability properties see [1], [2], [5], [6].

One of the notions that has had the greatest importance in the study of control systems is input-to-state stability (ISS). This concept was developed by Sontag in the late 1980s, and it becomes very useful for various branches of nonlinear control theory, such as, e.g., design of nonlinear observers [3], robust stabilization of nonlinear systems [4], etc. We say that a system is ISS if its behavior remain bounded when inputs are bounded, and goes to the equilibrium when inputs approach zero. The fact that the solutions of an ISS system are bounded under arbitrary bounded inputs makes ISS a very strong requirement in many applications. Hence, sometimes it is impossible to guarantee the ISS behavior of the closed loop system. That is why, some other relaxations of the ISS concept have been proposed following two main philosophies. The first one stands in a local version of ISS (namely local ISS [7]), which looks for ISS property when both the state and the input signal belong to a compact neighbourhood of the origin [8]. The second one is known as integral input-to-state stability (iISS [2]).

The iISS property was introduced and studied in [2]. A system displays the so-called iISS, if its trajectories are bounded when inputs have a finite energy. Applications of the iISS property can be found in [9], [10]. Every ISS system is necessarily iISS, but the converse is not true [2]. However, iISS property stills meaningful. It has been shown that ISS (i.e. iISS, local ISS) are equivalent to the existence of a smooth ISS (i.e. iISS, local ISS) Lyapunov function by Sontag and Wang in [11]. ISS framework was defined for different kinds of dynamics (differential inclusions, hybrid systems, discret-time systems, PDEs, etc.) and for multistable systems in [12], [13] and it was extended to multistable systems with delay [14], [15].

This work is also interested in the so-called strong iISS property. This concept was introduced in [5], [6]. Strong iISS property is introduced to offer an interesting compatibility between ISS and the generality of iISS. Hence, a system is strongly iISS, if it is iISS and ISS with respect to small inputs. This concept ensures that the states of any strongly iISS system is globally bounded as the $\mathcal{L}^{\infty}-$norm of the input is less than a specific constant, when it is greater than this constant we still may guarantee the iISS properties of the system. The strong iISS property apears in many nonlinear systems and it is useful in the analysis of interconnected systems (cascade structure [6]).

Since the main approach to establish ISS or iISS property consists in finding a corresponding Lyapunov function, and there is no method to a systematic design of such a function, then the research focus can be shifted on particular classes of systems. An example of such a class of dynamics is formed by homogeneous systems. The notion of homogeneity allows some local properties to be extended globally. Many results dealing with analysis and design have been proposed for homogeneous dynamical systems (see [16]–[23]). Some robust stability conditions have been presented for homogeneous dynamical systems (considering exogeneous disturbance), by giving sufficient conditions on the weight and degree of the homogeneity. In particular, if a system with external inputs is homogeneous and asymptotically stable without disturbance, then it is ISS or iISS (see [24]). It was also shown (see [25]–[27]) that if a system is homogeneous and stable, then there exists a homogeneous Lyapunov function. In this work, we will also investigate the finite-time convergence property. This concept has been introduced by Erugin 1951 and Zubov 1957, and studied by Roxin 1966, Korobov 1979 [28], [29] and the explanation for the abbreviation finite-time stability (FTS) by Bhat & Bernstein 2000 [18], [30]. To establish FTS property we will use Lyapunov function approach and homogeneity.

This paper is organized as follows. After introducing some definitions about robust stability and genralized homogeneity in Section II, we will investigate the robustness of homogeneous systems with respect to multiplicative perturbation and

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[1]Y. Braidiz and W. Perruquetti are with Ecole Centrale, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France, email: {youness.braidiz,wilfrid.perruquetti }@centralelille.fr.

[2]D. Efimov and A. Polyakov are with Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France and with Department of Control Systems and Informatics, University ITMO, 197101 Saint Petersburg, Russia. email: {denis.efimov, andrey.polyakov}@inria.fr.

the robustness of a system, which admits a homogeneous approximation (Section III). These results are applied in Section IV to the observation error dynamics problem that was defined in [31]. One of the main aims of this paper is to provide conditions to guarantee the finite-time convergence of the state of homogeneous system with multiplicative perturbation, when the disturbance is less than a determined constant.

Notation: $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers; $|\cdot|$ denotes the absolute value in $\mathbb{R}$ and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$ denotes the unit sphere in $\mathbb{R}^n$, $B(x_0, r) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$ denotes the open ball of radius $r > 0$ centered at a point x_0 and $\|A\|_{\mathcal{M}_{m,n}} = \sup_{x\in S} \|Ax\|$, $A \in \mathcal{M}_{m,n}$, where $\mathcal{M}_{m,n}$ is the set of all $m \times n-$matrices over the field of real numbers, it forms a vector space and $\|\cdot\|_{\mathcal{M}_{m,n}}$ denotes the matrix norm induced by $\|\cdot\|$. When $m = n$ we write $\mathcal{M}_n$ instead of $\mathcal{M}_{n,n}$. For a (Lebesgue) measurable function $d : \mathbb{R}_+ \longrightarrow \mathbb{R}^m$ define the norm $\|d\|_{[t_0,t_1)} = \text{ess} \sup_{t\in[t_0,t_1)} \|d(t)\|$, then $\|d\|_\infty = \|d\|_{[0,+\infty)}$ and the set of d with the property $\|d\|_\infty < +\infty$ we further denote as $\mathcal{L}_\infty$ (the set of essentially bounded measurable functions). A continuous function $\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing. The function $\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(.,t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_+$ and $\lim_{t\longrightarrow+\infty} \beta(s,t) = 0$ for each fixed $s \in \mathbb{R}_+$. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ belongs to the class $\mathcal{GKL}$ if $\beta(.,t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_+$, β is a strictly decreasing function of its second argument $t \in \mathbb{R}_+$ for any fixed first argument $s \in \mathbb{R}_+$ and $\beta(s,T) = 0$ for each fixed $s \in \mathbb{R}_+$ for some $0 \leq T < +\infty$. The notation $\langle DV(x), f(x)\rangle$ stands for the directional derivative of a continuously differentiable function V with respect to the vector field f evaluated at point x. $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ denotes the space of functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ which are smooth. We denote by $\mathcal{CL}^0(E,F)$ (respectively $\mathcal{CL}^k(E,F)$) the set of continuous functions on E, locally Lipschitz on $E \setminus \{0\}$ with value in F (respectively the set of continuous functions on E, $\mathcal{C}^k$ on $E \setminus \{0\}$ with value in F).

II. Preliminaries

A. Stability properties

In this subsection we give some definitions of robust stability (for more details see [24]) and FTS which will be investigated for a system with disturbances:

$$\dot{x}(t) = f(x(t), \delta(t)), \quad t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, and $\delta(t) \in \mathbb{R}^m$ is the external input, $\delta \in \mathcal{L}_\infty$. The vector field $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a locally Lipschitz (or Hölder) continuous function, $f(0,0) = 0$. For an initial condition $x_0 \in \mathbb{R}^n$ and input $\delta \in \mathcal{L}_\infty$, define the corresponding solutions by $x(t, x_0, \delta)$ for any $t \geq 0$ for which the solution exists.

1) Finite-time stability (FTS): Now we will give the definition of FTS which will be investigated in this paper. Let $\mathcal{V} \subseteq \mathbb{R}^n$ be a nonempty open neighborhood of the origin.

Definition 2.1: [30], [32] *Assume that the system* (1) *possesses unique solutions in forward time $t \geq 0$ for all initial conditions $x_0 \in \mathcal{V}$ except the origin for $\delta = 0$. The origin of the system* (1) *when $\delta = 0$ is FTS, if*

1) *- there exists a function $\alpha \in \mathcal{K}$ such that for all $x_0 \in \mathcal{V}$ we have $\|x(t, x_0, 0)\| \leq \alpha(\|x_0\|), \forall t \geq 0$.*
2) *- there exists a function $T : \mathcal{V} \longrightarrow \mathbb{R}_+$ such that if $x_0 \in \mathcal{V}$ then $\lim_{t\longrightarrow T(x_0)} x(t, x_0, 0) = 0$. T is called the settling-time of the system* (1).

In addition, if $\mathcal{V} = \mathbb{R}^n$, then the origin is a globally FTS (GFTS) for $\delta = 0$. If 1) *and* 2) *are satisfied for a given set of inputs δ we say that the system* (1) *is uniformlly FTS.*

2) Input-to-state stability: Now consider the definitions of ISS, iISS and finite-time ISS.

Definition 2.2: [1], [2], [33] *The system* (1) *is said to be*

- *ISS (resp., integral ISS (iISS)) iff there exist some functions $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ (resp., $\alpha \in \mathcal{K}_\infty, \gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$) such that for any initial state $x_0 \in \mathbb{R}^n$ and any $\delta \in \mathcal{L}_\infty$, the solution $x(t, x_0, \delta)$ exists for all $t \geq 0$ and satisfies*

$$\|x(t, x_0, \delta)\| \leq \beta(\|x_0\|, t) + \gamma(\|\delta\|_\infty) \tag{2}$$

$$\left(resp.,\ \alpha(\|x(t, x_0, \delta)\|) \leq \beta(\|x_0\|, t) + \int_0^t \gamma(\|\delta(s)\|)ds\right). \tag{3}$$

  If ISS property (resp., iISS property) holds with $\beta \in \mathcal{GKL}$, then (1) *is called finite-time ISS (resp., finite-time iISS).*
- *locally ISS (resp., locally finite-time ISS) iff there exist $\beta \in \mathcal{KL}$ (resp., $\beta \in \mathcal{GKL}$), $\gamma \in \mathcal{K}$ and $r > 0$ such that the inequality* (2) *holds $\forall x_0 \in B(0,r)$, $\forall \delta \in \mathcal{L}_\infty$: $\|\delta\|_\infty \leq r$ and $\forall t \geq 0$. If the estimate* (3) *holds for any $x_0 \in B(0,r)$, $\forall \delta \in \mathcal{L}_\infty$: $\|\delta\|_\infty \leq r$ and all $t \geq 0$, we say that the system* (1) *is locally iISS.*
- *ISS with respect to small input iff there exist a constant $R > 0$ and $\gamma \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and all $t \geq 0$, we have*

$$\|\delta\|_\infty < R \Longrightarrow \|x(t, x_0, \delta)\| \leq \beta(\|x_0\|, t) + \gamma(\|\delta\|_\infty).$$

Definition 2.3: [5], [6] *The system* (1) *is said to be strongly iISS iff it is both iISS and ISS with respect to small inputs.*

Definition 2.4: [23], [24] *A smooth function $V : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ is called*

- *an ISS Lyapunov function iff for all $x \in \mathbb{R}^n$, $\delta \in \mathbb{R}^m$ and some $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}$:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{4}$$

$$\langle DV(x), f(x, \delta)\rangle \leq -\alpha_3(\|x\|) + \gamma(\|\delta\|), \tag{5}$$

  such a function V is called ISS Lyapunov function with respect to $\mathcal{A} = \{x \in \mathbb{R}^n : \|x\| \leq A\}$ for some $A \geq 0$, if the inequality (5) *holds for all $x \in \mathbb{R}^n \setminus \mathcal{A}$ and it is iISS Lyapunov function if instead $\alpha_3 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a* *positive definite function. If there exists $\varepsilon > 0$ such that $\alpha_3(\|x\|) \geq cV(x)^{\alpha}$ for all $\|x\| \leq \varepsilon$ with $c > 0$ and $0 < \alpha < 1$, then V is a finite-time ISS (iISS) Lyapunov function,*

- *a local ISS Lyapunov function iff there exist some functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_{\infty}$, $\gamma \in \mathcal{K}$ and $r > 0$ such that the inequalities (4) and (5) holds for all $x \in B(0, r)$, $\forall \delta \in \mathbb{R}^m$ with $\|\delta\| \leq r$.*

Note that an ISS Lyapunov function can also satisfy the following equivalent condition for some $\chi \in \mathcal{K}$,

$$\|x\| \geq \chi(\|\delta\|) \Longrightarrow \langle DV(x), f(x, \delta)\rangle \leq -\alpha_3(\|x\|). \quad (6)$$

The following proposition shows how to investigate the FTS by using Lyapunov theory.

Proposition 2.1: [34] *Consider the system (1) with $\delta = 0$ and the uniqueness of solutions in forward time outside the origin. The origin of this system is FTS with a continuous settling-time function at the origin if and only if there exists a real number $c > 0, \alpha \in]0, 1[$ and a class $\mathcal{CL}^{\infty}$ Lyapunov function $V : \mathcal{V} \longrightarrow \mathbb{R}^+$ such that for all $x \in \mathcal{V}$*

$$\langle DV(x), f(x, 0)\rangle \leq -c[V(x)]^{\alpha}, \quad (7)$$

B. Generalized homogeneity

In control theory, homogeneity simplifies qualitative analysis of nonlinear dynamic systems. So that, it allows a local properties (e.g. local stability) to be extended globally using a property of the solutions (homogeneity is a special kind of Lie symmetry). In order to define this notion, let us introduce the notion of dilation group.

*Definition 2.5: A map $\boldsymbol{d} : \mathbb{R} \longrightarrow \mathcal{M}_n$ is called a **dilation** in $\mathbb{R}^n$ iff it satisfies*

- ***Group property:** $\boldsymbol{d}(0) = I_n$, $\boldsymbol{d}(t+s) = \boldsymbol{d}(t)\boldsymbol{d}(s)$, $t, s \in \mathbb{R}$.*
- ***Continuity property:** $\boldsymbol{d}$ is continuous, i.e. $\forall t > 0, \forall \varepsilon > 0, \exists \gamma > 0 : |s - t| < \gamma \Longrightarrow \|\boldsymbol{d}(s) - \boldsymbol{d}(t)\|_{\mathcal{M}_n} \leq \varepsilon$.*
- ***Limit property**: $\lim\limits_{s \longrightarrow -\infty} \|\boldsymbol{d}(s)x\| = 0$ & $\lim\limits_{s \longrightarrow +\infty} \|\boldsymbol{d}(s)x\| = +\infty$ uniformly on the unit sphere.*

Definition 2.6:
- *The dilation $\boldsymbol{d}$ is monotone in $\mathbb{R}^n$ if $\|\boldsymbol{d}(s)\|_{\mathcal{M}_n} \leq 1, \ \forall s \leq 0$*
- *The dilation $\boldsymbol{d}$ is strictly monotone in $\mathbb{R}^n$ if $\exists \beta > 0$ such that $\|\boldsymbol{d}(s)\|_{\mathcal{M}_n} \leq e^{\beta s}, \ \forall s \leq 0$.*

Property 2.1: The matrix $G_{\boldsymbol{d}} \in \mathcal{M}_n$ defined as $G_{\boldsymbol{d}} = \lim\limits_{s \longrightarrow 0} \dfrac{\boldsymbol{d}(s) - I_n}{s}$ is known as the generator of the group $\boldsymbol{d}(s)$ and satisfies the following properties: $\frac{d}{ds}\boldsymbol{d}(s) = G_{\boldsymbol{d}}\boldsymbol{d}(s) = \boldsymbol{d}(s)G_{\boldsymbol{d}}$ and $\boldsymbol{d}(s) = e^{G_{\boldsymbol{d}} s} = \sum\limits_{i=0}^{+\infty} \frac{s^i G_{\boldsymbol{d}}^i}{i!}$.

Let $S_{\mathbf{d}}(r) = \{x \in \mathbb{R}^n : \ \|\mathbf{d}(-\ln(r))x\| = 1\}, \ r > 0$ is called the homogeneous sphere of radius r. If $r = 1$ we write $S_{\mathbf{d}}$ instead of $S_{\mathbf{d}}(1)$. Now we will define the canonical homogeneous norm.

Definition 2.7: A continuous function $p : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ is said to be $\boldsymbol{d}$-homogeneous norm if $p(x) \longrightarrow 0$ as $x \longrightarrow 0$ and $p(\boldsymbol{d}(s)x) = e^s p(x) > 0$ for $x \in \mathbb{R}^n \setminus \{0\}$ and $s \in \mathbb{R}$.

*For monotone dilations the homogeneous norm $\| \cdot \|_{\boldsymbol{d}}$ is called the **canonical homogeneous norm***

$$\|x\|_{\boldsymbol{d}} = e^{s_x}, \ \text{where } s_x \in \mathbb{R} : \|\boldsymbol{d}(-s_x)x\| = 1. \quad (8)$$

Definition 2.8: A vector field $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ (resp., a function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$) is said to be $\boldsymbol{d}$-homogeneous of degree $\nu \in \mathbb{R}$ iff for all $s \in \mathbb{R}$ and all $x \in \mathbb{R}^n$ we have $e^{-\nu s}\boldsymbol{d}(-s)f(\boldsymbol{d}(s)x) = f(x)$, (resp., $e^{-\nu s}h(\boldsymbol{d}(s)) = h(x)$).

Homogeneity property was introduced also for Banach and Hilbert spaces, and it is given by a group of dilations [35], [36]. The following lemma provides a useful comparaison between homogeneous functions.

Lemma 2.1: [18] *Suppose that V_1 and V_2 are continuous real-valued functions on $\mathbb{R}^n$, $\boldsymbol{d}-$homogeneous of degrees $l_1 > 0$ and $l_2 > 0$, respectively, and V_1 is positive definite. Then, for every $x \in \mathbb{R}^n$, one has*

$$a_1[V_1(x)]^{\frac{l_2}{l_1}} \leq V_2(x) \leq a_2[V_1(x)]^{\frac{l_2}{l_1}}. \quad (9)$$

with $a_1 = \min\limits_{\{z; V_1(z)=1\}} V_2(z)$ and $a_2 = \max\limits_{\{z; V_1(z)=1\}} V_2(z)$.

The Lemma 2.1 has been proven in [18] for weighted homogeneous functions and we can use the same truck to prove it for $\mathbf{d}-$homogeneous functions V_1 and V_2.

In the following definition we introduce the notion of homogeneous approximation that has been used to study nonlinear control systems (see [37] and [38] for more details).

Definition 2.9: A function (resp., a vector field) f is said to be

- *homogeneous in the 0-limit with associated triple (ν_0, $\boldsymbol{d}$, f_0) iff*

$$\lim_{s \longrightarrow -\infty} \sup_{x \in K} \|e^{-\nu_0 s} f(\boldsymbol{d}(s)(x)) - f_0(x)\| = 0,$$

(resp., if $\lim\limits_{s \longrightarrow -\infty} \sup\limits_{x \in K} \|e^{-\nu_0 s}\boldsymbol{d}(-s)f(\boldsymbol{d}(s)(x)) - f_0(x)\| = 0$) for all compact subsets $K \subset \mathbb{R}^n \setminus \{0\}$.

- *homogeneous in the $\infty-$limit with associated triple (ν_∞, $\boldsymbol{d}_\infty$, f_∞) iff*

$$\lim_{s \longrightarrow +\infty} \sup_{x \in K} \|e^{-\nu_\infty s} f(\boldsymbol{d}(s)(x)) - f_\infty(x)\| = 0,$$

(resp., if $\lim\limits_{s \longrightarrow +\infty} \sup\limits_{x \in K} \|e^{-\nu_\infty s}\boldsymbol{d}(-s)f(\boldsymbol{d}(s)(x)) - f_\infty(x)\| = 0$) for all compact subsets $K \subset \mathbb{R}^n \setminus \{0\}$.

In the next section we will use the Lyapunov theory to show some robust stability results. The existence of a homogeneous Lyapunov function for a GAS homogeneous system was provided by Zubov in 1957, Rosier [25], in [39] by an explicit formula using the converse arguments, and by Polyakov [40], [41] by using weighted and generalized homogeneity.

III. Robustness analysis of a homogeneous system with respect to multiplicative perturbation

Robustness is the ability of the sensitivity of the system to be low to component variations. Robust methods aim to achieve robust performance and/or stability in the presence of bounded modelling errors. In this section we will study the robustness of the system (1) with the following assumptions:

- $\mathcal{H}_1 : f(x, \delta) = H(x)\,[b + \delta]\,, \ b \in \mathbb{R}^m \setminus \{0\}\,.$

- $\mathcal{H}_2$: $H : \mathbb{R}^n \longrightarrow \mathcal{M}_{n,m}$ is continuous and $x \longmapsto H(x)b$ is **d**-homogeneous with degree of homogeneity ν. Which mean that H is **d**-homogeneous.
- $\mathcal{H}_3$: The system $\dot{x} = f(x, 0)$ is globally asymptotically stable (GAS).

In particular, the hypothesis $\mathcal{H}_3$ means also that b does not belong to the kernel of $H(x),\ \forall x \in \mathbb{R} \setminus \{0\}$.

We first present the following theorem which shows some robustness properties of the system (1) under the previous assumptions ($\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$).

Theorem 3.1: Assume that the system (1) *satisfies the assumptions* $\mathcal{H}_1, \mathcal{H}_2$ *and* $\mathcal{H}_3$. *Then it is*

- *uniformly GAS for small inputs for any* $\nu \in \mathbb{R}$,
- *strongly finite-time iISS and uniformly globally FTS (GFTS) for sufficiently small inputs if* $\nu < 0$.

Theorem 3.1 shows that when the degree of homogeneity $\nu 0$, the dynamical system (1) is GFTS for all sufficiently small perturbations, and it is also iISS for every essential bounded input. This proves that we can guarantee finite-time convergence of the state of the system (1) even in the presence of disturbances.

Let $\mathbf{d}_1$ be a dilation which is defined from $\mathbb{R}$ to $\mathcal{M}_n$. The following theorem introduces some robust stability for a system with a vector field which admits a homogeneous approximation.

Theorem 3.2: Let $\delta \in \mathbb{R}^m$. *Assume that* $f(\cdot, \delta) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is homogeneous in 0-limit with associated triple* $(\nu_0, \boldsymbol{d}_1, f_0(\cdot, \delta))$ *where* f_0 *satisfies the assumptions* $\mathcal{H}_1, \mathcal{H}_2$ *and* $\mathcal{H}_3$. *Then, the system* (1) *is*

- *locally asymptotically stable (LAS) for small inputs for all* $\nu_0 \in \mathbb{R}$,
- *locally iISS and uniformly FTS for sufficiently small inputs when* $\nu_0 < 0$.

Here, in this theorem, we see that we can ensure some local finite-time stability for small inputs for systems which have a homogeneous approximation at the origin.

Theorem 3.3: If a function $f(\cdot, \delta) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, *is homogeneous in* $\infty-$*limit with respect to the triple* $(\nu_\infty, \boldsymbol{d}_1, f_\infty)$, *where the function* f_∞ *satisfies the assumptions* $\mathcal{H}_1, \mathcal{H}_2$ *and* $\mathcal{H}_3$. *Then, the system* (1) *is*

- *uniformly globally FTS with respect to* $\mathcal{A} = \{x \in \mathbb{R}^n : \|x\|_{\boldsymbol{d}} \leq A\}$ *for some* $A > 0$ *and small inputs for all* $\nu_\infty \in \mathbb{R}^n$,
- *iISS with respect to* $\mathcal{A}$, *if* $\nu_\infty < 0$.

If $\nu_\infty > 0$, *for small inputs the rate of convergence to the set* $\mathcal{A}$ *is uniform* (*independent on intial conditions*).

Theorem 3.3 prove the GFTS to a compact set for systems which have homogeneous approximation at infinity for every real degree of homogeneity of the approximation function (which is given by the assumptions $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$) and for small inputs.

Example 3.1: Consider the system (1) *with the vector field* f *given by*

$$f((x,y),(\delta_1,\delta_2)) = \begin{pmatrix} 0 & -yx^2 \\ 2x^3 + y^4 & -y^3 + x^4 \end{pmatrix} \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \right],$$

it is homogeneous in 0-limit with associated triple $(2, \boldsymbol{d}, f_0(\cdot, \delta))$ *with* $\boldsymbol{d}(s) = \begin{pmatrix} e^s & 0 \\ 0 & e^s \end{pmatrix}$, $\delta = (\delta_1, \delta_2)$ *and*

$$f_0((x,y),(\delta_1,\delta_2)) = \begin{pmatrix} 0 & -yx^2 \\ 2x^3 & -y^3 \end{pmatrix} \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \right].$$

For the homogeneous Lyapunov function

$$V(x,y) = \frac{1}{2}x^2 + \frac{1}{2}(x+y)^2 \tag{10}$$

we get

$$\langle DV(x,y), f_0((x,y),(0,0)) \rangle = -y^4. \tag{11}$$

It follows that the origin for the system $(\dot{x}, \dot{y}) = f_0((x,y),(0,0))$ *is GAS (using Lasalle invariance principle). Furthermore, asymptotic stability is preserved in the presence of any perturbation which does not change the approximating homogeneous function. Then by using the Theorem 3.2 the system* (1) *is LAS for small inputs.*

IV. Application

In the paper [31] a nonlinear finte-time observer is proposed that is relevant to secure communications [42], [43]. In particular, authors consider a nonlinear system of the form

$$\begin{aligned} \dot{z} &= Az + f(y,u), \\ y &= Cz, \end{aligned} \tag{12}$$

where $z \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ in known input and $y \in \mathbb{R}$ is the measured output, and

$$A = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & 0 & \dots & \ddots & 1 \\ a_n & 0 & \dots & \dots & 0 \end{pmatrix}, \tag{13}$$
$$C = \begin{pmatrix} 1, & 0, & \cdots, & 0 \end{pmatrix}.$$

An observer for this system is designed as

$$\dot{\hat{z}} = A \begin{pmatrix} z_1 \\ \hat{z}_2 \\ \vdots \\ \hat{z}_n \end{pmatrix} + f(y,u) - \begin{pmatrix} k_1|\hat{z}_1 - z_1|^{\alpha_1}\mathrm{sign}(\hat{z}_1 - z_1) \\ k_2|\hat{z}_1 - z_1|^{\alpha_2}\mathrm{sign}(\hat{z}_1 - z_1) \\ \vdots \\ k_n|\hat{z}_1 - z_1|^{\alpha_n}\mathrm{sign}(\hat{z}_1 - z_1) \end{pmatrix},$$

the paper [31] proposes guidelines to select α_i and $k_i, 1 < i \leq n$ such that the error $e = \hat{z}_1 - z_1$ tends to zero in finite-time.

Then the observation error dynamics is given by

$$\begin{cases} \dot{e}_1 &= e_2 - k_1|e_1|^{\alpha_1}\mathrm{sign}(e_1) \\ \dot{e}_2 &= e_3 - k_2|e_1|^{\alpha_2}\mathrm{sign}(e_1) \\ \vdots & \\ \dot{e}_n &= -k_n|e_1|^{\alpha_n}\mathrm{sign}(e_1) \end{cases} \tag{14}$$

with $\alpha_i > 0,\ 1 < i \leq n$. To guarantee the homogeneity of the system (14) we choose $\alpha_i = i\alpha - (i-1),\ 1 < i \leq n$

with $\alpha > 1 - \frac{1}{n-1}$ and n is the dimension of e with the dilation

$$\mathbf{d}(s) = \begin{pmatrix} e^s & 0 & \cdots & 0 \\ 0 & e^{\alpha s} & \ddots & \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{((n-1)\alpha-(n-2))s} \end{pmatrix} \tag{15}$$

For an illustration, in this section we consider the problem (14) with $n = 3$, we get

$$\begin{cases} \dot{x}_1 &= -k_1|x_1|^{\alpha}\mathrm{sign}(x_1) + x_2, \\ \dot{x}_2 &= -k_2|x_1|^{2\alpha-1}\mathrm{sign}(x_1) + x_3, \\ \dot{x}_3 &= -k_3|x_1|^{3\alpha-2}\mathrm{sign}(x_1). \end{cases} \tag{16}$$

Our goal is to study the same problem with some multiplicative disturbances

$$\begin{cases} \dot{x}_1 &= -\left(k_1 + L(t)\right)|x_1|^{\alpha}\mathrm{sign}(x_1) + x_2, \\ \dot{x}_2 &= -\left(k_2 + \frac{k_2}{k_1}L(t)\right)|x_1|^{2\alpha-1}\mathrm{sign}(x_1) + x_3, \\ \dot{x}_3 &= -\left(k_3 + \frac{k_3}{k_1}L(t)\right)|x_1|^{3\alpha-2}\mathrm{sign}(x_1), \end{cases} \tag{17}$$

where $L(t)$ represents an additional tuning of the gains k_1, k_2 and k_3 perfomed on-line or due to an auxiliary measurement information (usually the values of k_i are related with the amplitude of uncertainty to compensate).
This problem could be written as

$$\dot{x}(t) = H(x(t))\left[b + \delta(t)\right], \tag{18}$$

with

$$H(x) = \begin{pmatrix} -|x_1|^{\alpha}\mathrm{sign}(x_1) & x_2 & 0 \\ -\frac{k_2}{k_1}|x_1|^{2\alpha-1}\mathrm{sign}(x_1) & 0 & x_3 \\ -\frac{k_3}{k_1}|x_1|^{3\alpha-2}\mathrm{sign}(x_1) & 0 & 0 \end{pmatrix},$$

$b = \begin{pmatrix} k_1 \\ 1 \\ 1 \end{pmatrix}$ and $\delta(t) = \begin{pmatrix} L(t) \\ 0 \\ 0 \end{pmatrix}$. The function H is $\mathbf{d}$-homogeneous with degree of homogeneity $\nu = \alpha - 1$ and

$$\mathbf{d}(s) = \begin{pmatrix} e^s & 0 & 0 \\ 0 & e^{\alpha s} & 0 \\ 0 & 0 & e^{(2\alpha-1)s} \end{pmatrix}.$$

The system (16) is GAS. So here we will apply Theorem 3.1 to conclude that the system (17) is uniformly GAS for small inputs for any $\alpha > 0$ and it is strongly iISS and uniformly GFTS for sufficiently small inputs if $1 > \alpha > 0$. The gains of the observer have been set as follows: $k_1 = 1000, k_2 = 240, k_3 = 24$ the Figures 1 and 2 show the boundedness of the state of the system (17) for different degree α and different essentially bounded disturbances.

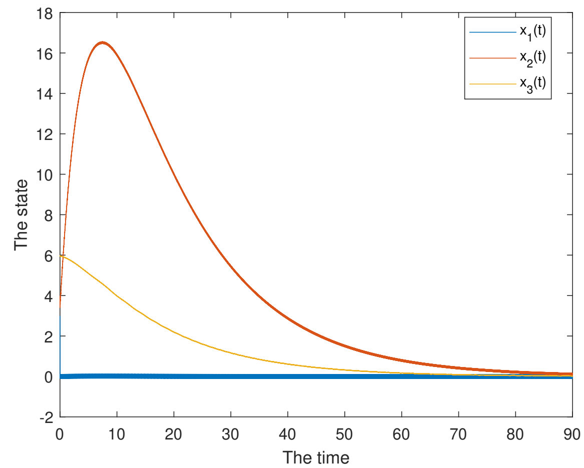

Fig. 1. The solutions of the system (17) with the initial condition $(x_0, y_0, z_0) = (3, 4, 6)$, $L(t) = 4 + \cos(t)$ and $\alpha = 0.7$.

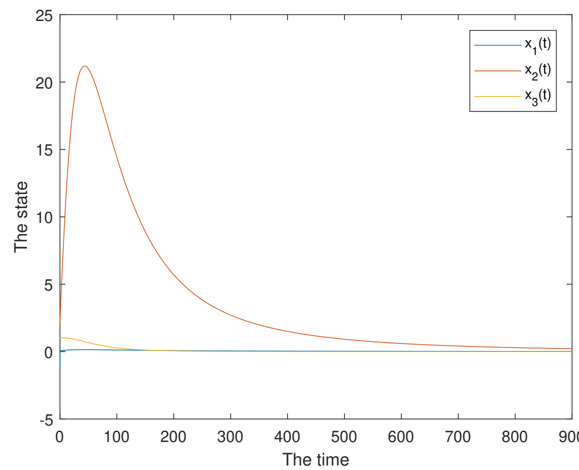

Fig. 2. The solutions of the system (17) with the initial condition $(x_0, y_0, z_0) = (-1, 2, 1)$, $L(t) = \frac{t}{t^2+1}$ and $\alpha = 2$.

V. Conclusion

In this paper, we studied the problem of robust stability of homogeneous systems with respect to multiplicative perturbations. Some results are established for the systems admitting homogeneous approximations at the origin and at infinity. The efficiency and practicality of the obtained conditions are demonstrated by considering a homogeneous observer with the gains dependent on functions of time, i.e. on additional measured information or adaptive tuning. Simulation results and academic examples are included for illustration.

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